

CHARACTERIZATION OF AMENABILITY BY A FACTORIZATION PROPERTY OF THE GROUP VON NEUMANN ALGEBRA

DENIS POULIN

ABSTRACT. We show that the amenability of a locally compact group G is equivalent to a factorization property of $VN(G)$ which is given by $VN(G) = \langle VN(G)^*VN(G) \rangle$. This answer partially two problems proposed by Z. Hu and M. Neufang in their article *Distinguishing properties of Arens irregularity*, Proc. Amer. Math. Soc. 137 (2009), no. 5, 1753–1761

It is known that the amenability of a locally compact group G can be characterized in many different ways. For example $A(G)$ has a bounded approximate identity ([Lep]; [He, Theorem 6, p. 129]), $A(G)$ factorizes weakly, i.e. $A(G)$ is the linear span of $A(G) \cdot A(G)$ [Lo, Proposition 2, p. 138], or $A(G)$ is closed in $M(A(G), A(G))$, its multiplier algebra [Lo2, Theorem 1]. It is also known that $A(G)$ is unital if and only if G is compact which is equivalent to $VN(G) = UCB(\hat{G})$. This last equality can be formulated for a Banach algebra A by $A^* = A^*A$. It seems possible to believe that if we relax the factorization property of the dual, we could characterize the existence of a BAI for $A(G)$, which is equivalent to the amenability of G by Leptin's theorem. More precisely, in this note, we prove that G is amenable if and only if the linear span $VN(G)^*VN(G)$, denoted by $\langle VN(G)^*VN(G) \rangle$, is equal to $VN(G)$. With this characterization of amenability for locally compact groups, we give a partial answer to the following problems proposed by Z. Hu and M. Neufang in [HN].

Problem 1 : Does $\overline{\langle VN(G)^*\square VN(G) \rangle} = VN(G)$ imply that G is amenable?

In this article, we made a stronger assumption than only the norm density of $\langle VN(G)^*\square VN(G) \rangle$ in $VN(G)$. However, with our assumption, we obtain a positive answer to this problem.

Another problem proposed in the same article of Z. Hu and M. Neufang is similar to the previous one. We stated here.

Problem 2 : If $VN(G)$ factors over $B_\rho(G)$, i.e., $\overline{\langle B_\rho(G)\square VN(G) \rangle} = VN(G)$, is G amenable?

By [Hu, Corollary 3.1], $\overline{\langle B_\rho(G)\square VN(G) \rangle} = \overline{\langle VN(G)^*\square VN(G) \rangle}$ if G is discrete.

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Thus, with Theorem 3, if G is discrete such that $\langle B_\rho(G)VN(G) \rangle = \langle VN(G)^* \square VN(G) \rangle$ and closed, then $\langle B_\rho(G) \square VN(G) \rangle = VN(G)$ implies that G is amenable.

Our argument will use the theory of abstract Segal algebras. The reader is referred to [Bu], [Bu2], [Le] and [Le2] to know more about this subject. For the definition and properties of the Fourier algebra $A(G)$ and its dual $V(G)$, the group von Neumann algebra, we refer to [E]. Also, to fix our notation, we recall the definitions of the Arens products. Let A be a Banach algebra. For $m, n \in A^{**}$, $f \in A^*$, $a, b \in A$,

$$\langle m \square n, f \rangle = \langle m, n \square f \rangle,$$

where $\langle n \square f, a \rangle = \langle n, f \square a \rangle$ and $\langle f \square a, b \rangle = \langle f, ab \rangle$. Similarly, using the right A -module structure of A on A^* , we have that

$$\langle m \triangle n, f \rangle = \langle n, f \triangle m \rangle,$$

where $\langle f \triangle m, a \rangle = \langle m, a \triangle f \rangle$ and $\langle a \triangle f, b \rangle = \langle f, ba \rangle$. For a Banach algebra A and a A -bimodule X , we denote by $\langle XA \rangle$ and $\langle AX \rangle$ the linear span of XA and AX respectively.

We present the main tool of this note which is an improvement of [P, Theorem 3.3.3].

Theorem 1. *Let A be a faithful Banach algebra. Let B be a proper right (left) abstract Segal algebra in A . Then $B^* \neq \langle B^* \triangle B^{**} \rangle$ ($B^* \neq \langle B^{**} \square B^* \rangle$).*

Proof. We will proceed by contradiction. Let $B^* = \langle B^* \triangle B^{**} \rangle$. We first prove that each element of $\langle B^* \triangle B^{**} \rangle$ as a unique extension on A . From there, we will conclude that the norm of A and B are equivalent on B which is a contradiction. Let $m_i \in B^{**}$, $i = 1, \dots, n$. By Goldstein's theorem, for each i , there exists a bounded net $m_{i,\alpha}$ in B such that $m_{i,\alpha} \xrightarrow{w^*} m_i$ in B^{**} . Let $f_i \in B^*$, and $b \in B$. Observe that

$$\begin{aligned} \left| \left\langle \sum_{i=1}^n f_i \triangle m_i, b \right\rangle \right| &= \left| \sum_{i=1}^n \langle m_i \triangle b, f_i \rangle \right| \\ &= \left| \sum_{i=1}^n \langle m_i \square b, f_i \rangle \right| \\ &= \left| \sum_{i=1}^n \lim_{\alpha} \langle m_{i,\alpha} b, f_i \rangle \right| \\ &\leq \sum_{i=1}^n C_i \sup_{\alpha} \|m_{i,\alpha}\|_B \|b\|_A \|f_i\|_{B^*} \\ &\leq M \|b\|_A \end{aligned}$$

This shows that $\sum_{i=1}^n f_i \triangle m_i$ is bounded on B with the norm of A . Since B is dense in A , then there exists a unique extension of $\sum_{i=1}^n f_i \triangle m_i$ on A . Let $\iota : B \rightarrow A$ be the natural inclusion map. Then ι^* is injective by the density of B , and for $h \in A^*$, $\iota^*(h) = h|_B$. Let us prove that ι^* is surjective. Let $f \in B^*$. Since $B^* = \langle B^* \triangle B^{**} \rangle$, then there are $f_i \in B^*$ and $m_i \in B^{**}$ such that $f = \sum_{i=1}^n f_i \triangle m_i$. Now, if we apply ι^* on the extension of f , we get f . From this, we deduce that ι^* and ι^{**} are

isomorphisms. Hence, there exist non-zero constants D_1 and D_2 such that

$$D_1 \|m\|_{B^{**}} \leq \|\iota^{**}(m)\|_{A^{**}} \leq D_2 \|m\|_{B^{**}}$$

for all $m \in B^{**}$. Since, $\iota^{**}(b) = \iota(b) = b$ for all $b \in B$, we get from the previous inequalities that the norm of A and the norm of B are equivalent on B , which contradicts the fact that B is a proper abstract Segal algebra. \square

Corollary 2. *Let A be a right, respectively left, faithful Banach algebra. If $A^* = \langle A^* \triangle A^{**} \rangle$, respectively $A^* = \langle A^{**} \square A^* \rangle$, then the norm of A is equivalent to its right, respectively left, multiplier algebra.*

Proof. We do a proof by contraposition. Suppose that the norm of A is not norm equivalent to the norm of its right multiplier algebra $RM(A)$. Then A is a right abstract Segal algebra in its closure in $RM(A)$. By Theorem 1, $A^* \neq \langle A^* \triangle A^{**} \rangle$. \square

We mention here that the equivalence of the norm of a Banach algebra A with its right or left multiplier algebra does not imply that A has a bounded approximate identity (see [W, Example 5]).

Using Theorem 1, we give a positive answer of problem 1 of [HN] if $\langle VN(G)^* \square VN(G) \rangle$ is already closed and not only norm dense. Note that the commutativity of $A(G)$ implies that $VN(G) \triangle VN(G)^* = VN(G)^* \square VN(G)$.

Theorem 3. *Let G be a locally compact group. Then G is amenable if and only if $VN(G) = \langle VN(G)^* \square VN(G) \rangle$.*

Proof. If G is amenable, then by Leptin's theorem, $A(G)$ has a bounded approximate identity and so $VN(G) = VN(G)^* \square VN(G)$. If $VN(G) = \langle VN(G)^* \square VN(G) \rangle$, then by Corollary 2, the norm of $A(G)$ and its multiplier algebra are equivalent. Thus, by [Lo2, Theorem 1], G is amenable. \square

It is interested to compare this result on factorization of $VN(G)$ with the classical factorization property treated by A. Lau and A. Ülger in [LU], i.e., $A^* = A^*A$ for a Banach algebra A with a BAI. In the case of the Fourier algebra, for any locally compact group G , the equality $VN(G) = VN(G) \triangle A(G) = UCB(\hat{G})$ implies that $A(G)$ is unital. This is not the case in general for Banach algebra. Take for example, $A = K(X)$, where X is a non-reflexive Banach space such that X^* has the bounded approximation property and the Radon-Nikodym property. By [P, Corollary 4.2.12], $A^* = A^*A$, but $K(X)$ is not unital. However, Lau-Ülger asked a question after [LU, Theorem 2.6] which was made as a conjecture by the author [P, Conjecture 5.3.5], we state it here:

Conjecture 4. *There is no faithful infinite-dimensional non-unital weakly sequentially complete Banach algebra A with a BAI such that $A^* = A^*A$.*

This conjecture is true for many Banach algebras like strongly Arens irregular Banach algebras, Arens regular Banach algebras, Banach algebras which are ideals in their biduals. To get a more complete list with proof of each case, see [P, Theorem 5.3.6]. Theorem 3 would suggest that for weakly sequentially complete Banach algebra, the factorization $A^* = \langle A^* \triangle A^{**} \rangle$ captures the existence of a BAI. We can reformulate this by saying that the factorization $A^* = \langle A^* \triangle A^{**} \rangle$ can characterize the co-amenability of a locally compact quantum group \mathcal{G} .

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DEPARTMENT OF MATHEMATICS, CARLETON UNIVERSITY, OTTAWA, ONTARIO, CANADA

Current address: Department of Mathematics, 1125 Colonel By Drive, Ottawa, Ontario, K1S 5B6, Canada

E-mail address: dpoulin@connect.carleton.ca